Prime numbers and the structures of the universe of codes

Andrea Gregori[†]

Abstract

We discuss how, within the framework of a previously introduced theoretical scenario proposing the equivalence of the physical quantum relativistic universe and the universe of logical/numerical codes, prime numbers play a key role as base blocks of physical structures. They constitute a kind of "free states" of a representation of physics alternative to the usual field-theoretical one, based on plane waves. As such, they can be useful in addressing problems which in the traditional approach are difficult to solve, or even to phrase. In particular, we discuss the relation between the scaling of certain physical quantities, and the distribution of prime numbers.

[†]e-mail: gregori.research@gmail.com

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1 Introduction

In Ref. [1] we have presented an updated discussion of a theoretical scenario which can be viewed as a way of ordering the whole of information in its most generic sense. The basic formulation is given in terms of mappings from the set of natural numbers to a vector product of discrete spaces. In this space of logical structures, or strings of information, we have introduced a time ordering using the natural ordering given by the inclusion of sets. Through the interpretation of logical codes in terms of distributions of energy along a target space, we have shown how this space leads, in the continuum limit, to a universe with the physical and geometrical properties of the universe we live in, with a three-dimensional space governed by a quantum-relativistic physics. The physical universe is given by the superposition of all the configurations, of any space dimensionality, at a given total amount of energy, which plays also the role of time, or age of the universe. Three dimensional space arises as the dominant dimensionality. The basic expression is the sum over all the possible energy configurations, weighted by their entropy (the (relative) weight being given by the volume of their combinatorial group) in $\Psi(E)$, the space of all the configurations (that is, of all the codes, or logical structures) with a fixed total amount of energy, E:

$$\mathcal{Z}(E) \, = \, \sum_{\Psi(E)} \mathrm{e}^{S(\Psi(E))} \, , \tag{1.1}$$

where $S(\Psi)$ is the entropy of the configuration Ψ in the phase space $\{\Psi\}$, related to the volume of occupation in the phase space, $W(\Psi)$, in the usual way: $S = \log W$. This sum can be considered as the "partition function", or the functional generating all the observables, of the theory. The dynamics is intrinsic in 1.1, which means that the time evolution is uniquely given by the entropy-weighted sum: at any time $\mathcal{T} \sim E$ the universe, and therefore also any subregion/subsystem, is given by a staple of configurations, weighted by their entropy in the phase space of all the configurations that correspond to that given total energy E, or equivalently age $\mathcal{T}$, of the universe. Any type of "force" or is therefore entropic by definition. The infinite number of configurations sums up to produce, in any observable quantity one can define in the three dimensional space, a smearing which corresponds to the Heisenberg uncertainty.

In the previous works we have considered the limit to the continuum, in order to recover the ordinary description of physics. Traditionally, the basic bricks of the description of the physical world are in fact the plane-wave free asymptotic states. Their interaction is dealt with as a perturbation. This approach proves to be successful in the description of weak forces (weak and electroweak), as well as in the case of "large-scale", classical gravitation (although excluding the cosmological scale of the evolution of the universe, in which case the small quantum gravity effects sum up on the long distance and large time elapse). Our approach however provides us with a non-perturbative description of the universe, in which the actual universe results from the superposition of configurations weighted according to their statistical weight in the phase space of all the configurations. This weight corresponds to the volume of their combinatorial group, i.e. to the number of ways they can be formed, and is in turn related to the frequency with which they therefore do occur. In this approach, a relevant concept is therefore that of factorization of the weight of a configuration into prime factors, that we interpret as corresponding to the weight of the elementary structures of this universe. These structures have in principle nothing to do with the traditional physical elementary structures, such as for instance the elementary particles. Therefore, when it is a matter of describing a free electron, it is still convenient to switch to its quantum mechanical description as a free wave. But there are cases in which our "logical" elementary structures are more appropriate for a truly non-perturbative description, at least for the investigation of certain properties. We will see how this will allow us to understand certain similarities in the structure of the universe at different scales, and to derive the scaling behaviour of the weak and strong coupling.

2 Prime numbers and complexity of structures

It is a common observation that, in first approximation, the universe seems to reproduce its shapes at different scales. For instance, a planet surrounded by its satellites is a kind of miniature-version of a solar system. All this depends on the properties of the gravitational force, of course. But it is also true that, although in a very loose way, it is not completely wrong to imagine the atom as a small solar system. In first approximation, this appears to be due to the fact that also the electric force behaves in a similar way to the gravitational one, both at the classical level of Coulomb-like expression of the potential, and at a field-theoretical level, being both photon and graviton massless fields. But here we want to understand *why* the physical world is ruled by forces that in first approximation behave in a similar way, reproducing similar structures at different scales.

In our scenario, the universe, and therefore any physical system, is given by the superposition of an infinite number of configurations, each one with a different weight. If we want to look at the scale properties in order to see whether and why certain structures and shapes are roughly reproduced at different scales, we must first of all consider the average over the staple of configurations, i.e. the mean value of the geometry, contributing to form the universe at a certain scale, and then also mod out by the structures at lower scales. This last operation is required by the fact that, when for instance we compare a planet and its satellites with the solar system, we neglect the fact that certain elements of the solar system, namely certain planets, have themselves in turn the structure of small solar systems, and so on.

We want to obtain the number of elementary structures around a time/energy scale N. According to [1], at any energy scale N the most entropic configuration is the three-sphere of radius N. Its weight scales as $\exp N^2$ times a factor depending on the total volume of space, and a trivial factor $N!$, common to all the configurations at energy/time N, which in our discussion is always implicitly factored out. This factor accounts for the number of possibilities of placing the center of the three-sphere along a space of finite extension, whereas $\exp N^2$, the part of the weight depending on the intrinsic symmetry group of the sphere, has to be intended as the appropriate natural integer whose size scales as the exponential of the square of the radius: although we use the expression $\exp N^2$, here we are indeed always speaking of an integer number. As discussed in Ref. [1], in this setup one works always in a space regularized by a cut-off, to be eventually removed, which sets the volume of space and the number of dimensions to finite values. Under these conditions, as long as the cut-off sets a target volume much larger than the one of the sphere, the extra factor is almost the same for all the configurations with a volume close to the one of the three-sphere, and can be factored-out. Taking into account the cut-off becomes relevant for the very sparse configurations, in which the units of energy are distributed along a very large volume, much larger than the one of the three-sphere. On the other hand, as it has been discussed in Ref. [1], the weight of these configurations is much lower than the one of the three-sphere, which alone weights more than the sum of all the other configurations [1]. In our analysis, we can therefore *normalize* all the weights dividing by the extra-factor of the three-sphere, so that the weight of the three-sphere is simply $\exp N^2$. This will introduce non-integer weights, but since we are interested in the scaling properties, what counts here is the relative scaling of

[1]We can also safely restrict our considerations to three dimensions, because the weights of the spheres at different dimensionalities, which are anyway the most entropic configurations for each dimension, are exponentially suppressed and therefore contribute to corrections of much lower order.

subsets of numbers, and this can be investigated independently on the normalization we choose. The error due to the cut-off can be made arbitrarily small. In the limit of infinite volume of the target space, the volume to be factored out becomes the same for all the configurations. To better say, the distance between the actual weight W, taken out of the overall volume factor, and the closest integer number, $n(W)$, vanishes in this limit: $|W - n(W)| < \mathcal{O}(1/V)$, where V is the volume of the target space (not to be confused with the volume of the three-sphere, $\sim N^3$, which corresponds to the volume of the classical geometry of the universe), so that $|W - n(W)| \stackrel{V \to \infty}{\longrightarrow} 0$.

For what we have just discussed, at any physical energy scale N we can associate an integer n of approximate size $\sim \exp N^2$. The quantity of interest for us is the number of primes around n, $\Delta n / \Delta N$: this precisely indicates the number of independent, basic structures, around the chosen scale, neglecting higher or lower scales. In order to simplify the computation, instead of the finite interval we consider the derivative, which gives us the increment in the number of structures per increment of the scale. Consider the approximate formula giving the number of prime numbers up to the integer n, which, according to the theorem of primes, is the more and more exactly satisfied the larger and larger is the size of n:

$$\pi(n) \approx \frac{n}{\ln n} \, . \tag{2.1}$$

By inserting $n = \exp N^2$, and taking the derivative with respect to N, we obtain:

$$\frac{d\pi(n(N))}{dN} = \frac{d}{dN} \frac{\mathrm{e}^{N^2}}{N^2} = \frac{2N\mathrm{e}^{N^2}}{N^2} - 2\frac{\mathrm{e}^{N^2}}{N^3} \, . \tag{2.2}$$

In order to compare the behaviour at different scales we must then normalize the increments of our differential expression dividing by the scale N itself, obtaining:

$$\frac{d\pi(n(N))}{d\ln N} = \frac{1}{N}\frac{d\pi(n(N))}{dN} \approx \frac{2\mathrm{e}^{N^2}}{N^2} - 2\frac{\mathrm{e}^{N^2}}{N^4} \, . \tag{2.3}$$

We now mod out the number of structures at the lower scale, by dividing by $\pi(n(N))$, finally obtaining the expression we were looking for:

$$\frac{d\ln \pi(n(N))}{d\ln N} \approx 2 - \frac{2}{N^2} \, , \tag{2.4}$$

where we used the symbol $\approx$ in order to make clear that this is only an approximated expression, obtained by considering just the most entropic configuration. In first approximation, the r.h.s. of expression 2.4 is a constant. This tells us that, roughly, the world shows up with similar structures at different scales. Roughly speaking, one could say that, if one forgets quantum corrections (i.e. the contribution of the rest of the staple of configurations out of the classical one), "an atom is like a solar system", thereby justifying the Bohr planetary-like approximation of the atom. The second term in the r.h.s. of 2.4 comes from the logarithmic factor, which characterises the distribution of primes, singling them out of the whole set of natural numbers. It gives a $1/N^2$ correction, that looks negligible at large N. However, this correction is, depending on the scale, precisely of the order either of the quantum corrections, or of the corrections introduced in the classical geometry by matter clusters (observe also that the energy density of the universe scales like $1/N^2$). As we are going to discuss in the next section, this term can be considered an "interaction" term, that tells about the strength of medium and large range forces. Its decreasing behaviour tells us that at larger scales the world tends to become more "simple" in the sense of more classical and flat.

3 The scaling of couplings

We want here to see how knowing the distribution of prime vs non-prime numbers allows us to derive certain scaling properties of the couplings. In this theoretical framework, a coupling is a volume in the phase space of the geometric configurations of the universe: it measures the weight of a transformation of particles. Along the evolution of the universe couplings scale therefore basically like ratios of masses. However, physics is more complex than just direct transitions from particle A to particle B. Indeed, we distinguish between long range and short range forces, and between strong and weak forces. The turn point between these two is the unit of measure of all the scales: the Planck scale. The gravitational coupling has here per definition size 1 (see Ref. [2] for more detail). If the strength of the gravitational coupling is fixed, the strength of the electroweak coupling has been derived in Ref. [3] by going to a logarithmic representation of the physical world. As discussed in Ref. [2], this is the picture in which gravity is decoupled, and one can easily investigate the spectrum of the elementary particles. Once obtained the bare value of the electroweak coupling from a ratio of volumes at a certain age of the universe, the actual value at the appropriate physical scale has then been computed by running the bare value from the ground scale of masses via the typical logarithmic dependence on the scale the couplings possess in a perturbative field theory representation. This was justified by the fact that we wanted to obtain the value of the coupling at a given scale within a representation of physics corresponding to the usual perturbative one, the one in which elementary particles are considered.

In the light of the present analysis, we can get a further insight in what we are precisely doing when passing to a perturbative representation. Within the set of all possible configurations, a special role is played by those which have a weight that, once normalized to the three-sphere as above, is given by a prime number. They don't contain subsets corresponding to subgroups of their global symmetry group. As such, they must be viewed as "global" configurations: they describe the entire universe as a whole piece. We can test this interpretation by considering that, as compared to the other configurations, the "local" ones, the volume of their symmetry group should loose a factor corresponding to the volume of the universe. The weights of the global configurations must therefore roughly scale as $1/N^3$ of the weights of the local configurations. The heaviest local configuration is the three-sphere (the weight of the three sphere clearly cannot be a prime number, because the symmetry group of the sphere has subgroups, whose weight is an integer divisor of the weight of the sphere). As discussed in Ref. [1], the weight of the three-sphere is of the order of the entire sum of weights, that we indicate as $\mathcal{W}(N)$. If we indicate with $\mathcal{W}_{\rm global}(N)$ the total weight of the global configurations at time (or energy) N [2], we have that this scales approximately as:

$$\mathcal{W}_{\rm global}(N) \approx \frac{\mathcal{W}(N)}{N^3} \approx \frac{{\rm e}^{N^2}}{N^3}\,, \tag{3.1}$$

where we have approximated $\mathcal{W}(N) \approx {\rm e}^{N^2}$. Integrating over time, this gives a scaling:

$$\int_N \mathcal{W}_{\rm global}(n) \approx \frac{\mathcal{W}(N)}{N^2} \approx \frac{\mathcal{W}(N)}{\ln \mathcal{W}(N)}\,. \tag{3.2}$$

This expresses the relation between the total weight, up to the size $\mathcal{W}(N)$, of the global configurations, and the total weight of all the configurations. With the substitutions $\pi(n) \leftrightarrow \int_N \mathcal{W}_{\rm global}$ and $n \leftrightarrow \mathcal{W}(N)$, this is the same relation as between the number of primes and the natural numbers, expression 2.1. As previously discussed, as long as the regularization cut-off V is finite this is just a correspondence between the scaling behaviour of weights and sets of numbers. It becomes however an exact correspondence with the sets of natural and prime numbers in the limit in which the cut-off is removed by factorizing out V, i.e. the limit $V \to \infty$, when the weights become exactly integer numbers.

Decoupling gravity from the theory, and in particular separating the effects of gravity on the weak couplings, corresponds to looking only at the configurations that describe the long-range part of the interaction (masses, the "gravitational charges", correspond to localizable objects, and clearly belong to the local part of the set of configurations). The strength of the coupling is related to the weight of this subset of configurations. Looking at its running through the mass scales means considering the weight of this subset of configurations relative to the weight of the configurations building up the gravity part:

$$\alpha \leftrightarrow \frac{\int_M \mathcal{W}_{\rm global}(m)}{\mathcal{W}(M)} \approx \frac{1}{\ln \mathcal{W}(M)} \quad \Longrightarrow \alpha^{-1} \sim \ln \mu\,. \tag{3.3}$$

The actual energy scale μ is not the total energy of the universe, N: microscopic energy scales are a fraction of the total energy of the universe, produced by the fact that in the microscopic physics one looks just at a subregion of each geometric configuration. Rather than being the actual value of a coupling, expression 3.3

[2]The total weight is also the total number of ways the N units of energy can be distributed along space.

has to be understood as giving the scaling behaviour with respect to the energy scales. The logarithmic running of couplings catches the scaling of the long-range part of the interactions. It gives quite correctly the behaviour of the electroweak coupling through the energy scales *at fixed age of the universe* [3]. We get here also another way of understanding why in the perturbative theory masses are free parameters: to be rigorous, the perturbative theory doesn't know about masses, consistently with the fact that gravity is not normalizable with ordinary perturbative rules of field theory.

So far, our result can be viewed as an alternative derivation of a behaviour already expected from ordinary field theory, which predicts a logarithmic running of the coupling via renormalization flow. In that context, this is basically due to the fact that the coupling is dimensionless. The field theory approach becomes however problematic when dealing with the strong coupling. Indeed, in field theory the strong interaction can only be represented in a weak coupling phase, where it is formally written as a gauge interaction, like the electromagmetic interaction, with the only difference that it has a beta-function of opposite sign, implying the flow to strong coupling at low energy. In Ref. [3] we argued that, despite its being formally represented like the other gauge interactions, the strong coupling doesn't run logarithmically with the scale. Here we can have a better understanding of its behaviour. Even if it is depicted as a gauge interaction propagating through the exchange of massless bosons, in the "physical" phase, namely, at the strong coupling, the strong force involves only localized objects, and is not a "global", infinite-range interaction. The strength of the coupling is therefore related to the part of numbers which are not prime, with density $\sim 1 - \frac{1}{\ln n}$. As a consequence, its scaling is not logarithmic, but power-law ($\sim n^{\beta}$ for some exponent β), like different mass scales as compared to each other (see Ref. [3]). As discussed in Ref. [3], this behaviour is precisely what we need in order for the predictions obtained within this theoretical framework to match the experimental results. This also means that the representation of the colour interaction as a gauge interaction is an artifact; in fact, there is no physical phase in which the strong force is weak and can be well approximated by representing it as a perturbative gauge interaction. We can therefore summarize our results as:

$$\alpha_{\rm e.w.}^{-1} \sim \beta \ln \frac{\mu}{\mu_0} + \frac{1}{\alpha_{\rm e.w.}^0}\,, \tag{3.4}$$

$$\alpha_{\rm gr.} = 1\,, \tag{3.5}$$

$$\alpha_{\rm strong} \sim \alpha_{\rm strong}^0 \left(\frac{\mu}{\mu_0}\right)^{\beta}\,, \tag{3.6}$$

where we have substituted the discrete scale n with its approximated counterpart in the continuum, the (sub-Planckian) energy scale μ, $\alpha_{\rm e.w.}^0 \equiv 1/\ln \mu_0$ and $\alpha_{\rm strong}^0$ have a power-law dependence on the age of the universe $\mathcal{T}$: $\alpha_{\rm i}^0 \sim 1/\mathcal{T}^{p_{\rm i}}$ for appropriate exponents $0 < p_{\rm i} < 1$ (see Refs. [2] and [3] for more detail and the explicit computation of these parameters).

4 A derivation of the distribution of prime numbers

We discuss now how the distribution of the prime numbers among the natural numbers, expressed by 2.1, that we have so far taken as a known result of mathematics, can be independently obtained within our theoretical framework. We will also see that the conditions on its regularity have a clear physical meaning.

In the scenario corresponding to 1.1, at time/energy N the universe is given by a staple of configurations, the heaviest of which is the three-sphere. Besides the three-sphere, there is the contribution of a whole bunch of configurations with lower amount of symmetry, and lower weight. The order of the sum of all the weights is lower than the weight of the three-sphere alone. As discussed in Ref. [1], the actual weight in the phase space of all the configurations contains a factor depending on the volume of the target space, which serves as cut-off and is eventually removed, and a trivial factorial of the N units of energy, common to all the configurations at time N. If the second factor is always factored out, the volume-dependent factor in principle becomes the same for all the configurations only in the infinite target-space volume limit.

[3] The so-called weak interaction is a medium range interaction which consists of a "long range part", the pure coupling, which behaves, and scales, similarly to the electromagnetic coupling, and a suppressing mass term, which works as a kind of cut-off, so that the effective coupling is $\alpha_{\rm w}/M_W^2$. The scaling of $\alpha_{\rm w}$ is logarithmic.

On the other hand, in the light of the analysis of Ref. [1], in which it is shown that the configurations in three dimensions weight exponentially more than those in any other dimension, and that the physics of the effective universe is basically given by the staple of configurations which occupy a portion of target space volume of the same order as the one of the three-sphere, we can restrict our considerations to the subset of the most significant configurations, neglect those with mean energy density very small as compared to the one of the three-sphere (and therefore much more extended in space), and so neglect also the contribution to the weight that depends on the target space volume, that we factor out. Under these restrictions, and after these factorizations, the weight of the sphere is $\sim \exp N^2$, expression which has to be intended in the sense that the three-sphere has a weight given by a non-prime integer number of order (size) $\sim \exp N^2$, which is constructed by multiplying the weights (i.e. the group volumes) of all the symmetry groups of this maximally symmetric space. Since we are working in a discrete space, all these are finite discrete groups [4].

As we have seen in Ref. [1], the geometric series leading to the uncertainty relation of quantum mechanics is derived by considering the configurations we can obtain by displacing from the three-sphere energy cells one after the other by unit steps. When we move one energy unit by one step, we create a "hole" in the former position and a neighbouring peak of energy. In this way we break the full geometric symmetry of the sphere. The volume of the symmetry group is therefore reduced by a factor $\sim N^3$, the volume of the sphere. In Ref. [1], in order to compute the new weight of the configuration we considered that this move can equivalently be done on each one of the N unit-energy cells forming the sphere. Therefore, the weight of the configuration we obtain is N times larger:

$$W' \sim \frac{N}{N^3} \times \mathrm{e}^{N^2} . \tag{4.1}$$

Breaking the full geometric group of the sphere means that we have created a space which doesn't have subgroups. This means that, once taken out the multiplicity factor N, its *weight is a prime number*. A priori this is not obvious. If a weight is a prime number, certainly the symmetry group it corresponds to doesn't have subgroups. Otherwise, since we are talking of finite discrete groups and integer weights (group volumes), the existence of subgroups would imply factorizability of the weight. But also the reverse holds: if the symmetry doesn't have subgroups, the weight is a prime number. In order to understand how this is possible, we must have a closer look at how the configurations are constructed. When we build-up a sphere, we arrange energy units in such a way that we have a product of all the possible symmetries, corresponding to rotations in all the possible (discrete) angles and directions. Here one must pay attention to the fact that when we speak of product of circles we do not intend it in the sense of Cartesian product of geometric circles (a kind of hyper-torus, to speak) but as the product of symmetry groups isomorphic to the group of rotations of a certain number n of elements (the discrete groups C_n). The weight of a sphere grows *almost exponentially* with the square of the radius N because the sphere is formed by a product of a huge number of C_n groups. Any weight of a geometric configurations can trivially be viewed as the volume of a circular group. If the number is not prime, this means that this group will have subgroups [5]. Our point is to see the relation between subgroups of the C_W group, and the *geometric* subgroups of the geometry a configuration corresponds to. When we displace a unit cell from the sphere, we break *all* the *geometric* symmetries of the space. The only symmetry which remains is isomorphic to the group of rigid rotations of the whole weight in the phase space, $C_{W'/N}$, where W' is given as in 4.1. This symmetry is not a symmetry of the geometric space this weight corresponds to. If W'/N were not a prime-number, we would have some "circle" C_m, with m such that $W'/Nm \in \mathbb{N}$, that would remain as a surviving subgroup. This would either imply a factorization of the geometric space into a product of spaces (e.g. the product of two or more spheres), or the existence of some symmetry of the geometric space, possessing subgroups. In short, a prime number weight can only correspond to a space possessing a trivial symmetry under rigid rotation of the whole geometric space (the "universe") within the target space of the energy units. This symmetry is trivial in the sense that its multiplicity only contributes to building up the size of the weight, but does not correspond to a physical symmetry of what we call the universe, which, we recall, is not the target space of the units of energy, but the

[4] Here by "sphere" one has to intend the closest discretization of the smooth maximally symmetric space with energy density $\sim 1/N^2$.

[5] For instance, the group of the hexagon is C_6, and has the subgroups C_2 and C_3. The symmetry of a pentagon is instead C_5, and doesn't have subgroups.

geometric space built up by the distribution of energy, interpreted in geometric terms through the Einstein's equation [6].

Let us come back now to W'/N, with W' given in 4.1. It is a prime number, as is a sequence of prime numbers also the sequence similarly associated to the weight of all the configurations we obtain by displacing further cells from the three-sphere without partially reconstructing a smaller amount of symmetry among the displaced energy cells. As discussed in Ref. [1], at each such a step we produce a set of $\mathcal{O}(N)$ configurations of weight around an order $\sim \mathcal{O}(1/N^3)$ lower than the previous set of weights. The reduction in the weight of the configuration due to displacements of energy units is in fact a sort of "soft breaking", in which each step given by the displacement of a unit of energy contributes by a similar amount. The order of overall reduction is roughly obtained by multiplication of the single contributions, because multiplicative is the structure of the phase space of weights, and through multiplication of (weights of) C groups was built also (the weight of) the sphere. Once we break the symmetry, speaking of multiplication factors is however only an approximation: the word "multiplication factor" must not be taken literally, because once we break the symmetry, we don't have anymore a factorizable weight. These factors do not stay therefore in integer ratios, and they must be intended just as approximations allowing to understand the order of the overall reduction. A closer insight in what is going on is obtained by thinking at the configurations with displaced units of energy as formed by the superposition of the sphere with configurations of single displaced cells. We can figure out what is happening by representing the configuration with the unit of energy displaced from A to A', a unit of space aside, as the superposition of the sphere plus the configuration in which the energy unit is removed from A (this operation subtracts a certain amount of weight), plus the configuration in which the unit of energy is added in A'. In order to estimate the weight of these latter, we consider taking away a pair of units, AB, and adding then the pair $A'B$. Indeed, the choice of B is irrelevant: it is easy to see that there is no difference in weight between $[-(AB)+(A')B]$ and $[-(AC)+(A')C]$, because their weight only depends on the distance (AA') (one can equivalently think of averaging over all the possible sums $[-(AC)+(A')C]$). The weight of these configurations is simply the weight of a pair of units (from which it is intended that we must factor out the trivial combinatorial factor). There are N^3 "triangles" $(AA'B)$, with A and A' fixed, and we obtain:

$$
\begin{aligned}
W' &= \langle W_{(3)} + W(A') \rangle \\
&= \frac{1}{N^3}\left\{ \mathrm{e}^{N^2} + \sum_{1}^{N^3} [-W(AB) + W(A'B)] \right\} \\
&= \frac{\mathrm{e}^{N^2}}{N^3} + [-W(AB) + W(A'B)] \\
&= \frac{\mathrm{e}^{N^2}}{N^3} + \mathcal{O}(1)\,.
\end{aligned}
\tag{4.2}
$$

Since we can perform the displacement $(A \to A')$ with all the N units of energy of the sphere, we recover as a result the previously estimated suppression factor of order $1/N^2$.

When we displace a second energy unit from the sphere, $B \to B'$, the distance and position of B' relative to A', the previously displaced one, is no more irrelevant in determining the weight, because we start from a situation of already broken symmetry. Since in the phase space we have $\sim N^3$ (the volume of the sphere) positions in which to equivalently realize the configuration $(A'B')$, a normalization factor $1/(N^3)$ is needed, leading to a further $1/(N^3)$ suppression factor in front of W'. Analogously to the previous case, the weight of the subtracted and added configurations results easier to compute if we think of subtracting from the

[6] We can give therefore a recipe for "constructing" a prime number around a size $M \sim \exp N^2$: consider the maximal symmetric space you can build with N units of energy, to be considered as sources of curvature in the sense of the Einstein's equation. Compute the weight of this construction, i.e. count the number of ways you can place the energy units along a $d = 3$ discrete target space, factoring out the contribution dependent on the volume of the target space, i.e. the counting of the possible positions of the center of mass of the sphere, and a $N!$ trivial combinatorial factor (a factor that in our constructions is always implicit, and implicitly factored out from all the constructions at energy N). Consider the geometric interpretation of this space, i.e. as a three-sphere. Move one unit of energy along the sphere by one step. The weight of the new construction is a prime of order $1/N^3$ lower than M.

sphere, and then adding back, one more unit of energy C, and averaging over C. We have:

$$\begin{aligned} W'' &= \langle W' + W(A'B') \rangle \\ &\approxeq \frac{1}{N^3} \{ W' + \langle -W(A'B(C)) + W(A'B'(C)) \rangle \} \\ &\approxeq \frac{N}{(N^3)^2} \times \mathrm{e}^{N^2} + \mathcal{O}(1) \,. \end{aligned} \tag{4.3}$$

Notice that this time we loose a multiplicity factor N in front of the added term, $W(A'B')$, because now the weight of the configuration is sensitive to the relative position of A' and B': the more A' and B' are far apart, the less the configuration $(A'B')$ weights, because the lesser are the possibilities to place it within the same target volume [7]. The differences in the weights $W(A'B')$ and $W(A'B'')$ for different choices of second displaced energy unit, B' or B'', are of the order of the distance between B' and B'', as one can easily see by considering that these weights correspond to the number of possibilities to form, within the same target space, configurations with 2 (3 if we consider also the generic point C) units of energy at a certain distance from each other.

Similar considerations can be applied also to the further steps of reduction of symmetry, that therefore lead to a series of weight suppressions of order $\sim 1/N^2$. If we consider the first and second step in the symmetry reduction, we see that we have one prime weight of size W/N^3 and N prime weights of size $\sim W/N^6$. The *order* of the average spacing between prime weights is then $\mathcal{O}(N^3/N) \sim N^2$. More in general, the fact that the sequence of weights obtained by displacing unit energy cells from the sphere roughly scales in ratios $W_{\mathrm{i}}/W_{\mathrm{i+1}} \sim 1/N^2$ implies a similar *average* spacing of the prime weights. The reason is that at each such step we can view the overall weight of the prime configurations as given by a multiplicity factor times the average of the single prime numbers. This means that the mean separation between the prime numbers of the sequence is $\sim N^2 = \ln W$. The inverse of this spacing is the *density* of prime weights within all the weights. The number of prime weights up to the size W is therefore roughly given by:

$$\# \text{ of primes } \sim \frac{W}{\ln W} \,. \tag{4.4}$$

Since the density is not constant but scales logarithmically, to be more precise this relation should be better expressed through an integral. We can anyway calculate a bound on the error of this estimate. In order to do this, we consider that in this way we can obtain some $\mathcal{O}(N^2)$ configurations before falling into configurations at previous times/energy (that we don't want to over-count). This number is obtained by considering that $V_{\mathrm{eff.}}$, the volume at disposal for the deformation of the N units of energy, is approximately given by the difference of volumes of the three-spheres at radius N and $N-1$: $V_{\mathrm{eff.}} \sim [V(N) - V(N-1)] \sim \left[N^3 - (N-1)^3\right] \approx \mathcal{O}(N^2)$. We must also consider that as long as we displace units of energy we can also rearrange them in order to partially reconstruct a certain amount of symmetry. The heaviest weight we can obtain by partial symmetry restoration after having displaced all the N energy units corresponds to a product of two spheres of half the radius of the initial one:

$$W^{\times} \sim \mathrm{e}^{\left(\frac{N}{2}\right)^2} \times \mathrm{e}^{\left(\frac{N^2}{2}\right)^2} = \mathrm{e}^{\frac{N^2}{2}} = \sqrt{W_{(3)}} \,. \tag{4.5}$$

The effective density of new prime-numbers between time $N-1$ and time N is therefore the number of new prime-number-producing deformations, $\sim N^2$, divided by the range of weight computed on a multiplicative scale, $W_{(3)}/W^{\times} = W_{(3)}/\sqrt{W_{(3)}} = \sqrt{W_{(3)}}$. This gives $\sim N^2/\sqrt{W_{(3)}}$. Notice that, owing to our conservative approach, reflected in the choice of dividing by the heaviest weight, $W^{\times}$, this is certainly an underestimate of the density. The amount of new primes is the inverse of this quantity, namely:

$$\# \text{ of new primes } \lesssim \frac{W_{(3)}}{\frac{\sqrt{W_{(3)}}}{N^2}} = \sqrt{W_{(3)}} \times N^2 \,. \tag{4.6}$$

[7] From a classical point of view, this expresses the fact that we have a lower gradient of energy, that results in a lower degree of modification of the gravitational field, and therefore of the curvature. This implies a lower degree of breaking of symmetry.

This can be assumed to be an upper bound to the *error* Δ on the amount of primes:

$$\Delta \lesssim \sqrt{W_{(3)}} \times N^2 = \sqrt{W_{(3)}} \ln W_{(3)} \,. \tag{4.7}$$

All this, namely 4.4 and 4.7 have to be compared with a bound on the number of primes at scale W derived by H. von Koch assuming validity of the Riemann hypothesis on the zeroes of the zeta function [4]:

$$\left| \pi(W) - \frac{W}{\ln W} \right| \leq \sqrt{W} \ln W \,, \tag{4.8}$$

where $\pi(W)$ approximates the number of primes with size no larger than W. Indeed, H. von Koch has shown that this constraint, or, to be more precise, the corresponding condition on the logarithmic integral form, 4.8 with $\mathrm{Li}(n) = \int_2^n (\ln t)^{-1} dt$ at the place of $\pi(n)$, holds *if and only if* the Riemann hypothesis holds.

Notice that the "time" progress toward higher weights is exponential, namely, along the evolution $N \to N+1$ we do not span all the natural numbers because the increase skips several steps $\left(\mathrm{e}^{N^2} \stackrel{N \to N+1}{\longrightarrow} \mathrm{e}^{(N+1)^2} \right)$. This can give the impression of producing a too rough condition on the prime numbers. However, the fact that the "density" of primes scales logarithmically as compared to the natural numbers implies that, when progressing toward a higher scale of prime numbers, the minimal effective unit step on the natural numbers is exponential.

In deriving the weight of the configurations with two energy units displaced from the sphere, we have seen that we actually generate a whole bunch of prime-weight configurations, which are very close to each other. Indeed, by closer inspection one can see that, in moving the unit B to a nearby position B', we have indeed the possibility of moving B one step closer to A', or even one step farther away from A'. These two configurations differ by a two-unit-step length. Since the differences in the weights in the phase space of these type of configurations are given by their difference in length (i.e. by the difference in their "radius", which determines how many times they can be rearranged into a finite volume), we generate in this way a pair of prime-number weights differing by two. Considering then also the difference of weight between configurations obtained by moving different points $B_{\mathrm{i}} \to B'_{\mathrm{i}}$, $1 \leq \mathrm{i} \leq N$, we obtain a series of prime numbers differing by further integer numbers. Since the rate of growth of the pairs of primes (p, p') such that $p' - p = 2$ is an order N^2 lower than that of all primes, so is also their total number

$$\pi_2(W) \sim \frac{\pi(W)}{N^2} \sim \frac{W}{(\ln W)^2} \,. \tag{4.9}$$

This is quite reminiscent of the Hardy-Littlewood conjecture. At the present stage our analysis is however too rough in order to test also the precise constant that normalizes π_2, which is conjectured to be $2C_2$, with $C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2}$.

According to our analysis, the existence of prime numbers at any scale, and the condition on their distribution 4.8, are therefore tightly related to the existence of a quantum universe of any size. It is the necessary premise for the expansion of the universe. In section 3 we have seen that the configurations with weight corresponding to a prime number are related to the long-range interactions. Their existence is tightly related to the existence of a weakly coupled force allowing the interaction of matter via massless fields. On the other hand, since this interaction is the interaction of spinors, the building blocks of any kind of matter, we can say that the existence of prime numbers of any size is the necessary condition for the existence of our relativistic, quantum gravitational universe, and that also the long range forces are intrinsically of quantum nature. This last statement is not surprising, if one thinks that the Heisenberg's derivation of the Uncertainty Principle was precisely based on considerations about the properties of light. If the occurrence of prime numbers within the integers were not sufficiently regular, the evolution of the universe would have discontinuity steps. These seem to be excluded by the smooth expansion of the string representation of this scenario: at large N, the representation on the continuum gives us a universe compatible with a smooth cosmological expansion, driven by the propagation of the massless fields, photon and graviton. The bound 4.7 on the regularity of the distribution of primes is some kind of overestimate, that only expresses the necessary (and sufficient) condition in order to preserve this property under time evolution, or, equivalently, propagation of the horizon. This condition is equivalent to saying that at any time the universe must correspond to a relativistic quantum gravitational scenario. There is no "Hamiltonian" representation of this time evolution or equivalently of the evolution toward higher scales of the universe, and of integer numbers. From a classical point of view, the Hamiltonian that propagates this system would be the one derived from the Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{4.10}$$

At the quantum level, quantization of space and time destroys the grounds of a possible Lagrangian formulation of the problem of evolution along a time coordinate: the latter can no more be dealt with as a parameter but becomes a field. Indeed, in our formulation the (value of the) time is also the total energy of the system [8]. This could explain why any attempt to find an appropriate Hamiltonian formulation to inductively prove the Riemann hypothesis as a problem of "time" evolution has till now failed: because the Riemann problem is equivalent to a complete relativistic quantum gravity set. In the light of our analysis the existence of the physical scenario proposed in Refs. [1, 2, 3] seems to be deeply related, in fact arguably equivalent, to the validity of the Riemann hypothesis on the zeroes of the ζ-function.

5 Considerations about the uncertainty relation

In section 3 of Ref. [1] we have derived the Heisenberg uncertainty relation by investigating the error in the estimate of the total energy of the universe, $E \sim \mathcal{T}$, where $\mathcal{T}$ is the age of the universe, obtaining $\Delta E \sim 1/\mathcal{T}$. This is however a somehow trivial case, in which the duration of the experiment is the time of the existence of the universe itself, in turn proportional to the total energy of the universe. Therefore, this relation could have also been written as $\Delta E \sim 1/E$. We have nevertheless extended the result $\Delta E \sim 1/\mathcal{T}$ to the case of "local" measurements and experiments, in the form $\Delta \mathcal{E} \sim 1/\Delta t$, where now $\mathcal{E}$ and Δt are the energy involved in the experiment, and its duration. In the case of any system except the whole universe itself, i.e. in any local case, the uncertainty relation cannot alternatively be written in the form $\Delta \mathcal{E} \sim 1/\mathcal{E}$: the uncertainty is related to the duration of the experiment, not to the inverse of its total energy. We want here to get a better insight on this relation, in the light of what we have learned from the previous sections.

Any knowledge about the physical world we can get through an experiment is obtained by collecting information which is propagated to us through the subset of configurations building up the universe constituted by the geometric configurations whose weight, when normalized as in the previous sections, is a prime number. Only these configurations correspond in fact to "long range" interactions. Let us consider a measurement of energy. What we measure is not directly the absolute amount of energy, but time variations of entropy/geometry. Therefore, we never directly measure the energy density of space. We just *infer* the value of the energy density by collecting measurements of what we call matter, i.e. (energy) clusters of elementary particles, of which all we know comes through the photons emitted by their interactions. During the time Δt of an experiment, this information comes to us superposed to the staple of configurations of a light-radius $\Delta t = \Delta N$, of which we detect only the prime-number-weight-part, i.e. we perceive the world only through configurations whose maximal weight at time N is $W_{(3)}/N^2$, a factor $1/N^2$ lower than the weight of the ground geometry of the universe. This is the reason why during a time N we obtain an uncertainty of order $\Delta E \sim 1/N$: the ground contribution, the contribution that weights $W_{(3)}$, is decoupled from the measurement, and is only inferred in an indirect way.

This observation allows us to understand how does it happen that we can observe configurations with a geometry quite asymmetric and therefore extremely rare in the phase space, such as for instance the case

[8] On the string side, what ensures the consistency of the scenario at any scale is the existence of massless modes such as the graviton and the photon. These "stir" the horizon of the universe, which, in this theoretical framework, expands at the speed of light (see the discussion of section 3, Ref. [3] about the apparent acceleration of the expansion of the universe). Besides the already to be expected approximation introduced in passing from a discrete description to the continuum, in talking about massless modes of the string scenario representing the cosmological scenario natively introduced on the discrete there is a further approximation, due to the fact that at any finite time the space is curved. Pure plane wave solutions of the equations of motion are obtained in the approximation of infinitely extended space-time, therefore the error due to the finiteness of space is arguably of order $\sim 1/N^2$, the curvature of the three-sphere.

of high-temperature superconductors. In the computation of the symmetry breaking of the sphere we have seen that a small departure from the initial position (one-step move of one energy unit) reflects on all the symmetry subgroups of a configuration, and has a quite huge effect in reducing the weight, which is given by the product of the weights of all the subgroups. Therefore, even if each one of them is only slightly reduced, by an amount of say $\epsilon \sim \mathcal{O}(1/N)$, their product is reduced by a factor of order $\prod_N (1-\epsilon)^N \sim (1-\mathcal{O}(N/N))$. These configurations should therefore be almost exponentially suppressed as compared to the sphere. This seems to contradict the procedure adopted in Ref. [5], where, in order to derive the ratio of weights in the phase space of superconductors with different lattice structure, we just computed the ratio of their *local* symmetry groups, assuming that factorization of the rest of the geometry would have been a reasonable approximation. How can in that case a reduction of symmetry on one piece not affect the symmetry of the rest of the configuration, and in such a way that an even small change results in a huge weight suppression? The answer is that what we measure are not absolute quantities but variations in the geometry propagated to us by the prime-part of the configurations. When we inspect the physics of, say, a superconductor, it means that, within all the inputs we receive at any time from the world, we look just at the small piece of all what surrounds us that we identify as our experiment. It is like applying a selective, band-pass filter to the world around us. The actual weight of heavier configurations, as well as the absolute weight of the configurations we are looking at, don't have a real physical meaning, as long as we do not receive information from the parts which are factored, or filtered, out, i.e.. in practice, if these (parts of) configurations are not detected by our experiment. On the other hand, even so filtered our detections are not clean, but smeared by the uncertainty intrinsic in the detection process. This is due to the fact that for what concerns the information we collect, duration of the experiment and extension in space of the region which contributes to the information we collect are linearly related. One could think that the highest precision is attained with the shortest time. It is not so because the larger is the region from which we collect information (the longer the duration of the measurement), the better we can filter the information coming from just one part of the universe, i.e. the better we can distinguish our experiment from the rest of the world. At shortest (i.e. near Planck-size) time, indeed we have in principle a very precise information about the energy, but we don't have a really good shaped world: we don't see fine structures, simply because they do not exist. We need a certain size in order to see a finely shaped world. It is only at this stage that we can speak of band-pass filter-like observation. In this case, the information gets better "filtered" the longer is the measuring time, because higher gets the "signal-to-noise ratio".

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